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Algebraic coarsening in voter models with intermediate states

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Abstract
The introduction of intermediate states in the dynamics of the voter model modifies the ordering process and restores an effective surface tension. The logarithmic coarsening of the conventional voter model in two dimensions is eliminated in favour of an algebraic decay of the density of interfaces with time, compatible with model A dynamics at low temperatures. This phenomenon is addressed by deriving Langevin equations for the dynamics of appropriately defined continuous fields. These equations are analyzed using field theoretical arguments and by means of a recently proposed numerical technique for the integration of stochastic equations with multiplicative noise. We find good agreement with lattice simulations of the microscopic model.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The theory of phase ordering dynamics is a prominent tool to study and classify the non-equilibrium behavior of interacting particle systems [1]. Applications can be found in very different fields, ranging from the study of binary mixtures [2] over reaction–diffusion systems [3] to models of social dynamics and opinion spreading [4].

In many physical systems, phase ordering occurs through defect annihilation and domain coarsening, as it can be observed for example in the Ising model after a quench from the disordered phase into the ordered one at low temperatures [1]. Here, the typical length scale $L(t)$ of clusters of aligned spins grows algebraically with time, following the general behavior $L(t) \sim t^{1/z}$, where, e.g., $z = 2$ for non-conserved dynamics. The numerical value of the

dynamical exponent z for any given model here depends on universal features of the model, such as symmetries and conservation laws, and on the relevant external parameters.

The low-temperature ordering in Ising-like models exhibits another very general property: the existence of surface tension between domains, favouring the formation of smooth interfaces minimizing the curvature. This motivates the notion of *curvature-driven* coarsening. In presence of smooth interfaces, we also expect the density of interfaces $n(t)$ to be inversely proportional to the domain length $n(t) \sim L^{-1}$, thus decaying with similar algebraic law $n(t) \sim t^{-\delta}$ with $\delta = 1/z$.

Other systems exist in which coarsening takes place without surface tension [5]; the growth of clusters is there purely diffusive, driven by interfacial noise. A typical example is provided by the voter model [6], in which spins can take one of two possible states, and flip with a probability linearly proportional to the fraction of unlike neighbors. In dimension $d = 1$, the voter dynamics is the same as zero-temperature Glauber dynamics of the Ising model [7], but increasing space dimensionality the ordering slows down and becomes logarithmic in $d = 2$ ($n(t) \sim 1/\log(t)$), whereas an infinite system does not order for $d > d_c = 2$. The notion of logarithmic coarsening in two dimensions here refers to the behavior of the density of interfaces with time, whereas the dynamical exponent, relating to the correlation length and properties of the dynamical structure factor, is still found to be $z = 2$ [8]. This is due to the critical nature of the voter model, as unveiled using field theoretic approaches [9, 10]. The voter dynamics corresponds to a massless field theory without field renormalization (from which $z = 2$), but it admits renormalization of other quantities, including the density of interfaces $n(t)$. Exact renormalization group calculations predict a behavior that agrees perfectly with both analytical solutions and numerical simulations [11, 12].

Recent research in opinion dynamics and sociophysics has amplified the interest in these two distinct classes of non-equilibrium systems, and has prompted their use in the study of the collective behavior of social individuals [4]. For instance, voter-like models with multiple states have been used to model groups of political parties [13, 14], the spreading of linguistic conventions [15–17], while the introduction of counters [18] or inertia in the spin dynamics [19] have been used to describe the effect of learning and memory in social interactions. Interestingly, the mere introduction of an intermediate state, without any major modification of the voter dynamics itself, has been seen to be sufficient to avoid logarithmic coarsening in two dimensions, restoring an algebraic decay of the density of interfaces and an effective surface tension [15, 18]. As we will show below, the qualitative asymptotic behavior does not change if one allows for more than one intermediate state, provided that only two absorbing states are present and that they correspond to the extreme states.

A better understanding of this intriguing phenomenological picture can be obtained by regarding the voter model as a special point of a larger class of non-equilibrium systems with two absorbing states, exhibiting a $\mathbb{Z}_2$-symmetry, see [5, 20, 21]. Starting from this viewpoint, Al Hammal *et al* [22] have derived a simple phenomenological field theory, expressed in terms of a non-linear Langevin equation, which captures the essential characteristics of the dynamical behavior of these models, and which faithfully reproduces the logarithmic coarsening dynamics of the voter model.

The aim of our work is to provide a theoretical explanation for the restoration of curvature-driven coarsening by intermediate states, and to corroborate the microscopic view with a continuum approach able to clarify the link with known results on models in the so-called generalized voter (GV) class [5, 22]. In the following we show analytically that voter models with intermediate states (this includes e.g. naming games [15, 16]) can be regarded as members of this class, indirectly providing an example of non-trivial microscopic models belonging to

Table 1. Transition rates of the two models considered. The two-state (spin 1/2) model is the usual voter model, while the three-state model (spin 1) is that with an intermediate state. The quantities $f_{\pm}$, f_0 are the fractions of $\pm$, 0 spins among the four nearest neighbors of a given spin.

Model	Transition rates
Two-state model	$p_{\pm\to\mp} = f_{\mp}$
Three-state model	$p_{\pm\to 0} = f_{\mp} + \frac{f_0}{2}$, $p_{0\to\pm} = f_{\pm} + \frac{f_0}{2}$

the GV class. We will derive phenomenological equations for the three-state voter model, and describe how such equations are, within certain limits, useful to describe the dynamics of models with a larger number of intermediate states as well. These Langevin equations are studied both numerically and more formally using field theoretical arguments. We show that despite the presence of interfacial noise, in two dimensions the asymptotic behavior is compatible with the low (but non-zero) temperature regime of model A relaxational dynamics. We also show that, unlike in the Ising model, interface roughening is not expected to occur in higher dimensions.

2. Voter models with and without intermediate states

In the voter model [6], spins are arranged on a network with one spin at each node, and each spin can assume two possible values, e.g. ± 1. At each step of the microscopic dynamics one spin is selected at random, and subsequently assumes the state of one randomly chosen nearest neighbor. We introduce an intermediate zero-state, maintaining a dynamics similar to that of the voter model. One spin s and one of its neighbors s' are chosen randomly at each time-step. If the spin selected is in state $s = +1$ and the neighboring one in state $s' = -1$, then s assumes the value $s = 0$. Similarly for $s = -1$, $s' = 1$ the first spin will be set to $s = 0$. If $s = 1$ and $s' = 0$, then s will assume the state $s = 0$ with probability $1/2$ and remain at $s = 1$ otherwise. Similarly for $s = -1$, $s' = 0$. If $s = 0$, then s will adopt the state s' if $s' = \pm 1$, and will flip to either $s = 1$ or $s = -1$ with equal probability if $s' = 0$. The exact dynamical iteration prescriptions and transition rates for the voter model and the modified three-state model are summarized in table 1. Note that with these rules intermediate states do not persist in time, and the only possible absorbing states are those with either all spins up or all spins down. The condition that intermediate states are not absorbing is necessary to observe curvature-driven coarsening, otherwise the coarsening might happen without surface tension, like in the original voter model. In this work we will focus on models defined on d-dimensional regular lattices of lateral size L, and most of the numerical analysis is carried out in the two-dimensional case, where we impose periodic boundary conditions. Interaction occurs with the set of the four nearest neighbors of any given spin.

The two-state and three-state models behave in the same way in one dimension, since the presence of intermediate states only has the effect of renormalizing the width of the domain walls by a finite factor (see [16, 18] for similar considerations). In two dimensions the behavior of the model with intermediate states is instead very different from that of the standard voter model. This can be seen in figure 1, where we report the temporal evolution of the density of intermediate states for the three-state model. We note here that spins in the intermediate state are found at the interfaces of domains of up and down spins respectively, so that the density of intermediate states is a measure of the density of interfaces (or domain boundaries). In the standard two-dimensional voter model, the interface density is known to decrease logarithmically, while the modified (three-state) model shows algebraic decay, i.e.

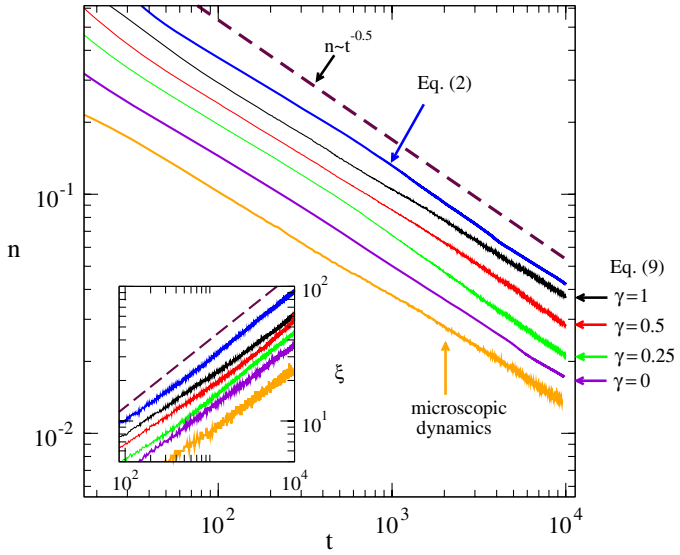

Figure 1. Density of intermediate states (main panel) for the three-state model. Results from one sample of a direct microscopic simulation of a system of size 2000^2 are shown, along with measurements from numerical integration of equations (2), where we have ignored the noise terms (see the text; system size is 500^2, averages over 10 runs with independent random initial conditions), and from equation (9) at $a = 1/2$, $D = 1$ and different noise strengths γ (curves at $\gamma > 0$ are obtained from systems of size 500^2, integration method as explained in the main text, averages over 10 samples are taken; results for $\gamma = 0$ are from Euler-forward integration, time-stepping of 0.05, 5 samples of size 1000^2). The inset shows the behavior of the correlation length ξ of the field ϕ (obtained from the circularly averaged correlation function). All curves have been rescaled by arbitrary constant factors for optical convenience. One unit of time corresponds to one update per spin on average.

$n(t) \sim t^{-\delta}$, where δ has been reported in the range of 0.45 to 0.5 [15]. The correlation length ξ, which we have computed from the two-point correlation function, grows with a similar algebraic law (see inset of figure 1). Both measures seem to suggest a dynamical exponent $z = 2$, even if from these numerical values the contribution of logarithmic corrections cannot be excluded. These were also the conclusion of previous studies [15, 18], and are corroborated by the marginality of voter dynamics in two dimensions. The point will be clarified in the following sections by means of a more theoretical analysis.

Figure 2 displays snapshots of the coarsening dynamics started from a disordered configuration (top), and during a 'droplet experiment' [5], in which a circular domain of up spins is left to evolve in a sea of down spins (bottom). In the pictures it is visible that despite the voter-like update rule, the three-state model develops an effective surface tension, as already noted for similar dynamics with intermediate states, such as the naming game model [16], the noise-reduced voter model [18], and models of bilingualism [15, 23].

To investigate the role of the interfacial noise clearly visible in the snapshots in figure 2, we have analyzed the roughness properties of the interface between two domains. Let us consider a flat interface between +1 and −1 spins and study its evolution in time under the dynamics of the model (see [24] for details of such experiments in the context of other models). Figure 3 displays the behavior of the average fluctuations of the width of the resulting interface. More precisely, if we call h_i the displacement in the direction perpendicular to the

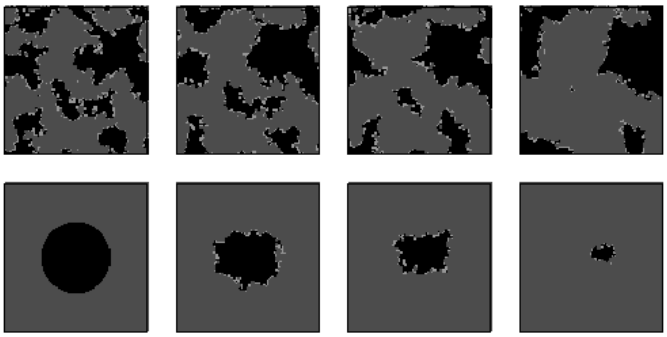

Figure 2. Snapshots of the time evolution of the model with intermediate states. Sites i with $s_i = -1, 0, 1$ correspond to black, white and grey respectively. Data are from simulations of the microscopic model, only a region of $N = 100^2$ sites is shown. The run in the upper row started from random initial conditions $s_i = \pm 1$, snapshots are taken after 100, 200, 400 and 800 sweeps over the system. Lower row shows evolution of a droplet (snapshots are taken at time $t = 0$ and then after 500, 1000 and 2000 sweeps).

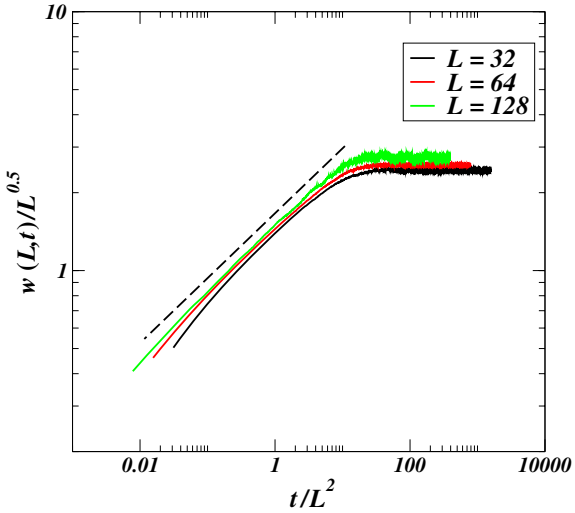

Figure 3. Data collapse for the interface width $w(L, t)$. The time is rescaled by L^z (having assumed $z = 2$) and the width is rescaled by L^α with $\alpha = 0.5$. This generates an initial power law growth t^β with $\beta \approx 0.25$ (dashed line). This rescaling is thus consistent with the Edwards–Wilkinson universality class of surface growth.

interface at position i (the latter is measured along the interface), the width is defined as $W = \left[\frac{1}{L}\sum_{i=1}^{L} h_i^2 - \left(\frac{1}{L}\sum_{i=1}^{L} h_i\right)^2\right]^{1/2}$, where L is the linear size of the interface. As the dynamics progresses W assumes a stationary value asymptotically. Considering systems of different sizes, we find that this saturation value W does not remain finite in infinite systems, but that instead the plateau value depends on the linear size of the system L. This indicates the presence of interface roughening phenomena. As in usual surface growth [24], the time evolution of the surface width exhibits two stages separated by a 'crossover' time $t_\times$. The initial growth follows a power law, $W(L, t) \sim t^\beta$, with growth exponent β; then for $t \gg t_\times$ the width saturates at $W_{\text{sat}}(L) \sim L^\alpha$ where α is the roughness exponent. Note that in the

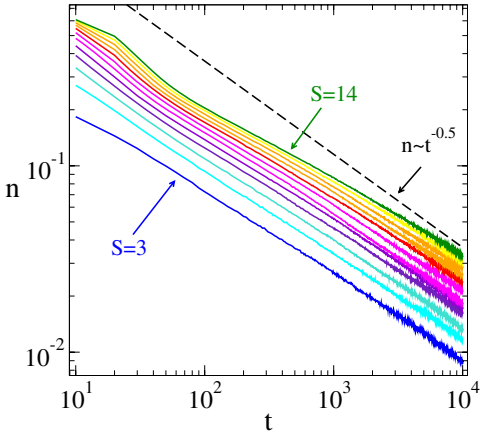

Figure 4. Coarsening of the model with multiple intermediate states. The plot shows the density of spins in intermediate states for the model at $S = 3, 4, \ldots, 14$ (solid lines from bottom to top). The dashed line is $n \sim t^{-0.5}$ and is shown as an optical guide only. Each curve represents data from a single run of the microscopic dynamics on a lattice of 1000^2 sites. One unit of time corresponds to one update per spin on average.

scaling theory of surface growth these exponents are related to the dynamic one by $z = \alpha/\beta$. Figure 3 verifies the Family–Vicsek scaling relation $W(L, t) \sim L^{\alpha} f(t/L^{z})$, with $\alpha = 0.5$ and $z = 2$. Accordingly we find $f(u) \sim u^{\beta}$ with $\beta \approx 0.25$ at small times. The good data collapse generated by rescaling, suggests that the interface dynamics in our model is compatible with the Edwards–Wilkinson (EW) universality class of surface growth [24].

In summary, the numerical results indicate that the introduction of an intermediate state makes the voter dynamics similar to the low-temperature ($T > 0$) scaling of model A [25], both as far as the domain growth and interface roughening are concerned.

Further generalization to multiple intermediate states has already been considered. For instance, in the naming game [16] a large number of different intermediate states can be created. These are however all equivalent with no ordering or hierarchy among them, therefore the effective behavior of the system is expected to be the same as for the three-state model. More interestingly we can define a voter model with non-equivalent intermediate states: let us consider $S \geqslant 2$ states in total, two of them being extremal absorbing states, and the remaining $S - 2$ ones being intermediate states. To implement this generalization let us label the states by $k = 1, \ldots, S$, with $k = 1$ and $k = S$ being the extremal ones. As before the model is considered on a two-dimensional lattice, and at each time-step one spin and one of its four nearest neighbors are chosen at random. Let $s \in \{1, \ldots, S\}$ be the state of the chosen spin, and s' that of the neighbor. Then set $p(s') = (s' - 1)/(S - 1)$. As a consequence of the interaction, spin in state s is then set into state $s + 1$ with probability $p(s')\Theta(S - s)$, and into state $s - 1$ with probability $(1 - p(s'))\Theta(s - 1)$. $\Theta(x)$ is here the step function with $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x \leqslant 0$. It ensures that spins are never modified to take values outside the set $\{1, \ldots, S\}$. As a consequence of the definition of $p(s')$, linearly increasing from 0 to 1 as s' increases from 1 to S, the extremal states 1 and S have maximally polarizing effects, whereas intermediate states induce flips of neighboring spins only at a probabilistic rate. For $S = 2$ the model reduces to the voter model, while for $S = 3$ we recover the three-state model as discussed above.

Results from simulations of this generalized model for several values of S are reported in figure 4, while some snapshots of the dynamics for the model at $S = 5$ are shown in figure 5.

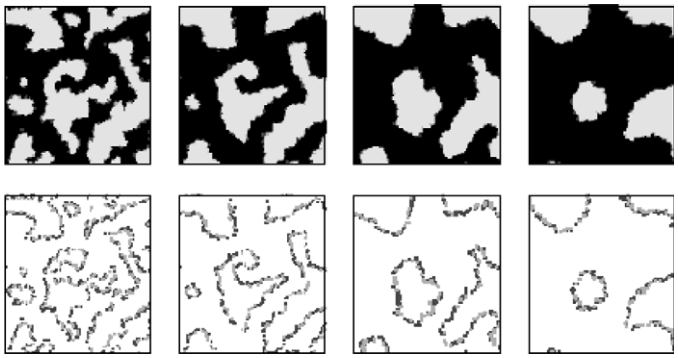

Figure 5. Snapshots of the coarsening dynamics of the model with five states. Data are from simulations of a system of size 300^2 sites, only a region of 100×100 spins is shown. Snapshots are taken after 200, 400, 800 and 1600 sweeps respectively (left to right). In the lower row we depict only spins in the intermediate states.

As seen in the figures, the coarsening of the model is curvature driven also when multiple intermediate states are present, and the density of intermediate states decays with an exponent close to the one already found for the three-state model almost independently of the number of allowed states S. In simulations with many intermediate states (S of the order of ten or more), interfaces can be expected to have a finite size (typically $\mathcal{O}(S)$). The definition of the measured quantity (density of spins in intermediate states) can hence be affected as the number of intermediate states is systematically increased, potentially limiting the accuracy of our numerical results and preventing one from a reliable measurement of the scaling exponent in simulations, even at comparably large lattice sizes (e.g. 1000^2). Still, results reported in figure 4 are consistent with algebraic coarsening at an exponent of 0.45, similar to that reported for the 3-state model in [15]. Numerical values of exponents obtained from fits in figure 4 are actually in the range of 0.42–0.48, depending on the time-window used for the fits, but with a tendency to increasing values of the exponent toward later phases of the simulations, so that it is fair to assume that once transients have fully died out, exponents in the range of 0.45–0.5 are to be expected. More precise numerical verification of the decay exponents in the later phases of the coarsening dynamics in voter models with many intermediate states is however still awaited.

Other definitions of multi-state models are of course possible. For instance one could consider a model with S states (again with an internal ordering between them) with two extremal absorbing states (1 and S), but with uniform transition probabilities among neighboring states (e.g. a given spin in state s may change to state $s + 1$ upon interaction with a spin in any state $s' > s$, with a rate which is independent of the specific state s'). We have tested such models (results not shown here) and have found algebraic domain growth with a coarsening exponent of about 0.45, so we conclude that the precise definition of the transition rates between internal states is immaterial as far as the growth of the characteristic length scale of the resulting domain structure is concerned.

3. Continuum description in the scaling regime

In order to complement simulation results and to obtain analytical insight into the dynamics of the voter model with intermediate states we develop a description of the three-state model in terms of continuous field variables. As the microscopic model is neither diluted nor endowed

with mobility, a standard derivation of continuous equations for coarse-grained degrees of freedom [26, 27] is not straightforward. For the same reasons and because of the fact that the model considered here is effectively a spin-1 model, an exact microscopic mapping to field theoretic methods using second quantization, e.g. following the lines of [28, 29], is difficult as well.

We will therefore proceed to derive approximate stochastic evolution equations directly from the microscopic rates, based on the assumption that fields are Gaussian random variables. Because of the nature of the derivation, these stochastic equations do not describe coarse-grained degrees of freedom, but microscopic ones and differ from those obtained by adding diffusion to mean-field like terms [26]. The macroscopic description becomes clear when the slow modes are identified, by applying power counting and neglecting higher-order terms in the momentum space. This approach is somewhat similar to the derivation of effective field theories in condensed matter spin models.

We start by considering a version of the model, defined as before on a two-dimensional regular lattice, but now endowed with Ω spins per site, and in which each spin can take any of the three states $\{-1, 0, 1\}$. The dynamics at each time-step then proceeds by selecting one site and one of the Ω spins at that site at random, and then executing the interaction with a randomly chosen spin at a randomly selected nearest neighbor site. The Ω copies of the system hence become cross-correlated.

In the limit $\Omega \to \infty$, deterministic equations for the evolution of average local quantities such as the average local magnetization ($\langle s_i \rangle$) and the average local density of zero states $\left(1 - \left\langle s_i^2 \right\rangle\right)$, can easily be obtained with the same rate equation approach applied by Krapivsky in the case of pure voter model [11, 12]. Averages $\langle \cdots \rangle$ are here understood as averages over the Ω copies of the microscopic dynamics, i.e. $\langle s_i \rangle$ is the average magnetization at site i resulting from the Ω spins at that site. In the limit of an infinite number of particles per site we replace $\langle s_i \rangle \to \overline{\phi}(x,t)$ and $1 - \left\langle s_i^2 \right\rangle \to \overline{\psi}(x,t)$. Using such continuous variables is appealing but dangerous as it may introduce ultraviolet divergencies as well as a certain level of arbitrariness in the definition of the continuum limit of local operators (like the lattice Laplacian). During the course of the calculations presented in this paper we will not consider a continuum limit in space, and the variable x is assumed to have the granularity of an arbitrary lattice spacing as in standard lattice field theories. We expect that the spatial continuum limit could be in principle considered appropriately defining a diffusion constant D.

If only a finite number Ω of spins per site is considered, then progress can be made starting from the master equation describing this ensemble of Ω interacting copies of the system. Let us label the state of spin i in copy k of the system by $s_i^{(k)}(t)$ at time t. We will denote averages over the Ω copies by $\widetilde{\phi}(x,t)$ in the following, i.e. we will use the substitution $\Omega^{-1} \sum_{k=1}^{\Omega} s_i^{(k)}(t) \to \widetilde{\phi}(x,t)$. Note that while overbars indicate averages in a system with an infinite number of spins per site, averages denoted by a tilde are over a finite number Ω of copies only. It is then easy to verify that a change in the state of spin i in one of the Ω samples corresponds to a transition of the type, $\widetilde{\phi}(x,t) \to \widetilde{\phi}(x,t) \pm \frac{1}{\Omega}$ and $\widetilde{\psi}(x,t) \to \widetilde{\psi}(x,t) \pm \frac{1}{\Omega}$ in the quantities describing the average over the Ω samples. This leads to a master equation describing the evolution of the functional probability distribution $\mathcal{P}[\{\widetilde{\phi}(x,t), \widetilde{\psi}(x,t)\}]$. Expanding to next-to-leading order in Ω^{-1}, we get the following multivariate functional Fokker–Planck equation for the functional probability distribution $\mathcal{P}$

$$\frac{\partial}{\partial t}\mathcal{P}(\widetilde{\phi},\widetilde{\psi};t) = \int \mathrm{d}^d x \left\{ -\frac{\delta}{\delta\widetilde{\phi}}\left[\left(\frac{\widetilde{\phi}\widetilde{\psi}}{2} + \frac{1+\widetilde{\psi}}{2}\nabla^2\widetilde{\phi}\right)\mathcal{P}(\widetilde{\phi},\widetilde{\psi};t)\right] \right.$$
$$\left. -\frac{\delta}{\delta\widetilde{\psi}}\left[\left(-\frac{3\widetilde{\psi}}{2} + \frac{1-\widetilde{\phi}^2}{2} - \frac{\widetilde{\phi}}{2}\nabla^2\widetilde{\phi}\right)\mathcal{P}(\widetilde{\phi},\widetilde{\psi};t)\right]\right.$$

$$+\frac{1}{2\Omega}\left(\frac{\delta^2}{\delta\widetilde{\phi}^2}+\frac{\delta^2}{\delta\widetilde{\psi}^2}\right)\left[\left(\frac{\widetilde{\psi}}{2}+\frac{1-\widetilde{\phi}^2}{2}-\frac{\widetilde{\phi}}{2}\nabla^2\widetilde{\phi}\right)\mathcal{P}(\widetilde{\phi},\widetilde{\psi};t)\right]$$
$$+\frac{1}{\Omega}\frac{\delta^2}{\delta\widetilde{\psi}\delta\widetilde{\phi}}\left[\left(\frac{1-3\widetilde{\psi}}{2}\nabla^2\widetilde{\phi}-\frac{3\widetilde{\psi}\widetilde{\phi}}{2}\right)\mathcal{P}(\widetilde{\phi},\widetilde{\psi};t)\right]\Bigg\}, \tag{1}$$

where we have rescaled time by a factor of Ω^{-1}, i.e. $t\to t/\Omega$. ∇^2 here is the lattice Laplacian, i.e. $\nabla^2\widetilde{\phi}(x,t)=\frac{1}{4}\sum_{y\in x}\{\widetilde{\phi}(y,t)-\widetilde{\phi}(x,t)\}$, where the sum over y extends over the four nearest neighbors of x. Noise terms in the corresponding Langevin equations for $\widetilde{\phi}$ and $\widetilde{\psi}$ are hence of order $\Omega^{-1/2}$. They reflect the fact that averages in a model with only a finite number Ω of spins per site still remain stochastic quantities. Stochasticity only vanishes if $\Omega\to\infty$, and deterministic mean-field equations for $\overline{\phi}$ and $\overline{\psi}$ are obtained (these would be equations (2) below, with the noise terms removed). Our approximation consists in describing the model with only one spin per site (i.e. the model introduced and discussed in the earlier sections) by the following two coupled Langevin equations

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x,t)&=\frac{1}{2}\phi(x,t)\psi(x,t)+\frac{1+\psi(x,t)}{2}\nabla^2\phi(x,t)+\eta(x,t),\\ \frac{\partial}{\partial t}\psi(x,t)&=-\frac{3}{2}\psi(x,t)+\frac{1-\phi(x,t)^2}{2}-\frac{\phi(x,t)}{2}\nabla^2\phi(x,t)+\xi(x,t),\end{aligned} \tag{2}$$

where $\eta(x,t)$ and $\xi(x,t)$ are Gaussian noise terms, with correlation given by

$$\begin{aligned}\langle\eta(x,t)\eta(x',t')\rangle&=\langle\xi(x,t)\xi(x',t')\rangle=\tfrac{1}{2}\{\psi(x,t)+1-\phi(x,t)^2\\ &\quad-\phi(x,t)\nabla^2\phi(x,t)\}\delta(x-x')\delta(t-t'),\\ \langle\eta(x,t)\xi(x',t')\rangle&=\langle\xi(x,t)\eta(x',t')\rangle=\tfrac{1}{2}\{-3\psi(x,t)\phi(x,t)\\ &\quad+(1-3\psi(x,t))\nabla^2\phi(x,t)\}\delta(x-x')\delta(t-t').\end{aligned} \tag{3}$$

We have therefore effectively disregarded higher-order terms in the expansion in Ω^{-1}, even if the model actually corresponds to $\Omega=1$. Nevertheless, as we will show below, equations (2) and (3) turn out to be a faithful mapping of the original dynamics of the three-state model onto a lattice theory with continuous fields at each grid point, capturing the essential feature of the original model. It is here worth stressing again that, similar to the voter model our microscopic dynamics do not allow for microscopic mobility of agents (e.g. hopping or interchange of neighboring particles), thus the diffusion-like terms in (2) are non-trivial features emergent from the microscopic model. In particular the diffusive terms in the deterministic parts of (2) are quadratic in the fields (e.g. $\psi\nabla^2\phi$ and $\phi\nabla^2\phi$). Similar structures have been found in the context of other spatial systems for example in [30].

A study of a version of our model including microscopic mobility can be performed along the lines of [31], here local mixing is ensured by scaling the hopping rate suitably with the system size, and an effective description in terms of continuous fields can be obtained by means of an expansion in the inverse system size. This leads to equations which are similar to but not identical to equations (2) and (3) [32]. More specifically one finds noise amplitudes of the form (3) with the terms proportional to $\nabla^2\phi$ removed and with an additional overall prefactor N^{-1} ($N=L^2$ being the system size). The deterministic terms take a form very similar to those reported in (2), but with decoupled diffusive terms ($\propto\nabla^2\phi,\nabla^2\psi$) instead of objects of the forms $\psi\nabla^2\phi$ and $\phi\nabla^2\phi$.

If we were to repeat the same procedure as above for the case of the pure voter model with two possible states (and without mobility or particle interchange), the resulting equation would be

$$\frac{\partial}{\partial t}\phi(x,t)=\nabla^2\phi(x,t)+\eta(x,t), \tag{4}$$

where $\langle\eta(x,t)\eta(x',t')\rangle = \{1-\phi(x,t)^2 - 2\phi(x,t)\nabla^2\phi(x,t)\}\delta(x-x')\delta(t-t')$. Apart from the diffusion term in the noise, equation (4) is identical to the field-theoretic equation proposed by Dickman *et al* for the voter model [33]. Naive power counting on dimensional grounds reveals that the diffusive terms involving Laplacians in the noise correlator are irrelevant in any dimension in the long wavelength limit. Long-time large-scale properties of a microscopic voter model are therefore well described by the effective Langevin equation

$$\frac{\partial}{\partial t}\phi(x,t) = D\nabla^2\phi(x,t) + \sqrt{1-\phi(x,t)^2}\eta(x,t), \tag{5}$$

with $\langle\eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$. Note that the same equation could have been obtained also by adding the diffusive term to a zero-dimensional mean-field Langevin structure [33]. Despite its validity, such coarse-grained derivation misses a clear microscopic justification.

Similarly, on the base of dimensional analysis we could neglect the terms proportional to $\nabla^2\phi$ in equation (3), providing slightly simplified noise terms with the same absorbing states.

As we will see, numerical integration of equations (2) and (3) is challenging (simple forward integration in discrete time-steps renders the field-dependent variances of the noise terms negative with finite probability), therefore it is useful to further reduce their complexity by means of some physically meaningful approximation.

Provided we focus on the asymptotic dynamics of the system, it is possible to reduce equations (2) and (3) to a single equation. We here exploit the observation that intermediate states cannot proliferate. Due to their immediate decay, zero states localize at the interfaces of large domains of spins in the states ±1. Hence the field $\psi(x,t)$ indicating the local presence of intermediate states is zero if $\phi(x,t)$ is close to -1 or 1, and non-zero only when $\phi(x,t)$ takes values far from the absorbing boundaries. However, the intermediate state is not an absorbing one, as spin flips toward the ±1 states occur. $\phi(x,t) \simeq 0$ may therefore correspond to a situation in which states ±1 are predominantly populated (at equal frequency), as opposed to the zero state. At a site x at which $\phi(x,t) = 0$, it is therefore fair to assume that the states ±1 occur at equal probability $(1-c)/2$ with $0 < c < 1$, and that the zero state is assumed at site x with some probability c. An approximation $\psi(x,t) \simeq c\{1-\phi(x,t)^2\}$ is thus justified and a single field ϕ can be expected to be sufficient to describe the behavior of domain walls driving the asymptotic coarsening. Within this approximation, and neglecting the small modulation of the diffusion constant induced by the term $\frac{1+c[1-\phi(x,t)^2]}{2}$ in front of the Laplacian in the first equation of (2), the system is described by the following equation for the local magnetization field ϕ,

$$\frac{\partial}{\partial t}\phi(x,t) = \frac{c}{2}[\phi(x,t) - \phi(x,t)^3] + \nabla^2\phi(x,t) + \sqrt{1-\phi(x,t)^2}\eta(x,t), \tag{6}$$

where η is the usual Gaussian noise, uncorrelated both in space and time and of unit variance. The first term on the right-hand side of equation (6), absent in the voter model, is the typical drift term related to a double-well potential in the sense of a Ginzburg–Landau model. It drives the system to relax toward the two minima of the potential at ±1, coinciding with the absorbing states of the three-state model.

In the previous section we have seen that the algebraic nature of the decay of the interface density and the corresponding coarsening exponent are not affected significantly when the number of intermediate states is increased, provided that we use a model in which interfaces remain compact (e.g. when the transition rules favor the convergence to the absorbing states). For the multi-state model proposed, we expect equation (6) to give a valid asymptotic effective description in the limit of continuous spins. To this end we study the limit in which the number of intermediate states S is large and in which we can approximate spin configurations

by a continuum profile. Specifically, consider $\rho = \frac{s-1}{S-1}$, where the spin state s takes values $s \in \{1, \ldots, S\}$. In the limit $S \gg 1$, ρ can then be thought of as a continuous degree of freedom (at each lattice point), taking values $\rho \in [0, 1]$. In order to recover a well-defined model it is then necessary to rescale the spatial dimension appropriately, viz. $x \to x/S - 1$. Uniform profiles at $\rho = 0$ or $\rho = 1$ are then absorbing states. In this limit the probability that the field ρ_i at site i increases (by an infinitesimal amount) at any given time-step is then linear in the value ρ_j of one randomly chosen neighbor with which interaction occurs.

The continuous representation provides a good long-time coarse-grained approximation for a model with large S in the coarsening regime, where $L(t)$ is much larger than the interface width. The coarse-grained nature of this representation allows one now to derive a phenomenological Langevin equation for the variable $\rho(x, t)$. Considering a Fokker–Planck equation for a zero-dimensional problem with variable ρ, which can increase by a given increment with probability ρ, and decrease by the same increment with probability $1 - \rho$, and imposing the presence of absorbing states at $\rho = 0$ and $\rho = 1$, we find

$$\frac{\partial}{\partial t} P(\rho, t) = \frac{\partial}{\partial \rho}[(1 - 2\rho)\rho(1 - \rho)P(\rho, t)] + \frac{\Gamma}{2}\frac{\partial^2}{\partial \rho^2}[\rho(1 - \rho)P(\rho, t)], \tag{7}$$

where Γ relates to the amount by which fields are incremented or reduced in the course of a microscopic step. The corresponding Langevin equation for the spatial dependent field $\rho(x, t)$ then reads

$$\frac{\partial \rho}{\partial t} = -(1 - 2\rho)\rho(1 - \rho) + D\nabla^2\rho + \sqrt{\Gamma\rho(1 - \rho)}\eta, \tag{8}$$

where we have added a diffusion-like term. The noise η is delta-correlated in time and space. The substitution $\phi(x, t) = 1 - 2\rho(x, t)$ in equation (8) yields a Langevin equation of the same form as equation (6). While no claim is made that this derivation is rigorous or exact, these simple arguments lead one to expect that the entire class of voter models with S intermediate states exhibits an asymptotic coarse-grained dynamics described by equation (6). Since equation (6) leads to algebraic coarsening as discussed below, this is consistent with our numerical findings for the multi-state model (see figure 4).

4. Analysis of the equations

4.1. Field theoretic analysis and relation to the GV class

In this section we analyze the field theoretic counterpart of this model, that can be generally identified starting from the equation

$$\frac{\partial}{\partial t}\phi(x, t) = a\phi(x, t) - a\phi(x, t)^3 + D\nabla^2\phi(x, t) + \sqrt{\gamma(1 - \phi(x, t)^2)}\eta(x, t), \tag{9}$$

i.e., a special case of the Langevin equation of the generalized voter class proposed by Al Hammal *et al* in [22] (one has $b = 0, a > 0$ in their notation). The fact that linear and cubic terms have the same coupling constant is necessary to enforce $\mathbb{Z}_2$-symmetry and has striking consequences for the corresponding field theory.

Using the response functional technique [28], we obtain the following effective action:

$$\mathcal{S}[\phi, \hat{\phi}] = \int \mathrm{d}^d x \, \mathrm{d}t[\hat{\phi}(\partial_t - a - D\nabla^2)\phi + a\hat{\phi}\phi^3 - \gamma\hat{\phi}^2 + \gamma\hat{\phi}^2\phi^2]. \tag{10}$$

Naive power counting reveals the canonical dimensions of the fields, $[\phi] = 1$ and $[\hat{\phi}] = k^d$ and that of coupling constants, $[a] = k^2$ and $[\gamma] = k^{2-d}$. The noise vertices become relevant for $d \leqslant 2$, whereas for $d > 2$ the process is not affected by multiplicative noise. In fact, the

critical theory ($a = 0$, i.e. voter model) can be renormalized exactly to all orders, leading to a complete characterization of the ordering process in any dimension, above and below $d_c = 2$ [9, 10].

For any $a > 0$, the system is in the broken symmetry phase with spontaneous magnetization and minima in ± 1. In analogy with model A below the critical temperature, the dynamical field theory should be governed by the strong coupling fixed point $a \to \infty$ [1, 25]. This can be verified by developing momentum shell renormalization for the coupling a. Under rescaling of lengths and times, $x \to bx$, $t \to b^z t$, one considers the transformations $D \to b^{z-2} D$, $a \to b^z a$ and $\gamma \to b^{z-d}\gamma$. Imposing scale invariance for the system's fluctuations, as required by coarsening regime, we get $z = 2$ and the tree-level flow equations $\frac{\mathrm{d}a}{\mathrm{d}\ell} = 2a$, and $\frac{\mathrm{d}\gamma}{\mathrm{d}\ell} = (2-d)\gamma$, where $b \equiv e^{\ell}$. These simple renormalisation group (RG) equations hence predict a strong coupling fixed point for a, playing the same role as the zero-temperature fixed point does in the Ising model. This means that in the region $a > 0$ we do not expect corrections to the mean-field dynamical exponent $z = 2$, so that this exponent, characterising the growth of the correlation length during coarsening, is expected to be accurate in any dimension. This is also true for $a = 0$ since the propagator does not acquire corrections. The density of interfaces on the other hand follows the general scaling $n(t) \sim 1/L(t) \sim t^{-1/z}$ in any dimension only for $a > 0$, whereas it is modified to a logarithmic behavior for the critical (voter) theory ($a = 0$). We have here disregarded corrections to the tree-level RG equations and have neglected higher-order diagrams. The absence of loop corrections for $a > 0$ can be seen to be justified by performing a mapping of this model on branching annihilation random walks (BARW) by rapidity inversion [29, 34]. Our model here corresponds to unphysical BARW with negative reaction rates. However, following [29], loop corrections shift the critical value at $a = 0$ to some $a_c < 0$, whereas our discussion concerns the absorbing phase at $a > 0$. Note that region $a < 0$ does not correspond to any microscopic model obtained adding intermediate states to the voter model, but presents a non-trivial competition between the two absorbing states at ± 1 and the minimum of the potential at zero [22]. For this case, a complete understanding of the RG diagram is still lacking [29, 34, 35].

The fact that noise flows to zero for $d > 2$ and to the strong coupling fixed point for $d < 2$ indicates that roughening phenomena may be present below the upper critical dimension $d_c = 2$, but that they are absent in $d > 2$. In dimensions higher than two, the system therefore orders like a zero-temperature Ising model. The behavior at $d = 2$ can be understood by mapping the field theory on the dynamics of an interface using standard methods [24]. Consider the Hamiltonian formulation for the equilibrium problem in absence of multiplicative noise $\mathcal{H} = \int \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right] \mathrm{d}^d x$, where $V(\phi) = \frac{a}{4}(1-\phi^2)^2$, and define the interface position as $z = h(y,t)$, with y being the $d-1$ dimensional space coordinate specified by defining a preferred direction z for the interface movement (i.e. $\vec{x} = (\vec{y}, \vec{z})$). Neglecting the multiplicative noise, it is a zero-temperature Ising model, so the interfaces will be mainly driven by diffusive forces [36]. On the other hand, the multiplicative noise term behaves like additive noise at the interface (since $\phi = 0$ near the interface), resulting in the following Edwards–Wilkinson (EW) type of equation,

$$\frac{\partial h(y,t)}{\partial t} = \tilde{D}\nabla_{\perp}^2 h(y,t) + \eta(y,t) \tag{11}$$

with $\langle \eta(y,t)\eta(y',t')\rangle = \gamma\delta(t-t')\delta^{d-1}(y-y')$. Therefore the interface roughening in $d \leqslant 2$ is described by the EW universality class in $d_{\mathrm{int}} = d-1$ dimensions. In $d_{\mathrm{int}} = 1$, EW universality class predicts roughening exponents $\alpha = 1 - d_{\mathrm{int}}/2 = 1/2$ and $\beta = \alpha/z = 1/4$, consistent with the behavior of the interface width W measured for our model in simulations (see figure 3).

Interestingly, we could also apply an infinitesimal magnetic field in order to break the $\mathbb{Z}_2$-symmetry, then expanding the fields around one of the absorbing states (e.g. $\phi = \phi_0 - \phi'$ with $\phi_0 = 1$) [37]. In this case, the resulting action $\mathcal{S}'$ for the perturbation field ϕ' reads

$$\mathcal{S}'[\phi', \hat{\phi}'] = \int \mathrm{d}^d x \, \mathrm{d}t [\hat{\phi}'(\partial_t + 2a - D\nabla^2)\phi' - 3a\hat{\phi}'\phi'^2 + 2\gamma\hat{\phi}'^2\phi' - \gamma\hat{\phi}'^2\phi'^2], \tag{12}$$

in which we have neglected higher-order terms in ϕ'.

Equation (12) is similar to the action of directed percolation (DP) [3], therefore it is convenient to put it in the standard DP form [38] by rescaling the fields. Let us define $\hat{\zeta} = \sqrt{3a/2\gamma}\hat{\phi}'$ and $\zeta = \sqrt{2\gamma/3a}\phi'$, after a bit of algebra we get

$$\mathcal{S}'[\zeta, \hat{\zeta}] = \int \mathrm{d}^d x \, \mathrm{d}t [\hat{\zeta}(\partial_t + 2a - D\nabla^2)\zeta - u(\zeta - \hat{\zeta})\hat{\zeta}\zeta - \gamma\hat{\zeta}^2\zeta^2], \tag{13}$$

where $u = \sqrt{6a\gamma}$. Like in DP, the pure noise coupling γ can be neglected with respect to u, since $[u] = k^{2-d/2}$ and $[\gamma/u] = k^{-d/2}$, thus the effective action becomes

$$\mathcal{S}_{\mathrm{eff}}[\zeta, \hat{\zeta}] = \int \mathrm{d}^d x \, \mathrm{d}t [\hat{\zeta}(\partial_t + 2a - D\nabla^2)\zeta - u(\zeta - \hat{\zeta})\hat{\zeta}\zeta]. \tag{14}$$

It is easy to check that one loop correction to the propagator shifts the bare critical point ($a = 0$) outside the region of interest (i.e. toward negative values $a_R < 0$). In other words, when we apply a small magnetic field to our system, the corresponding field theory should be that of directed percolation in the absorbing or active phase depending on the sign of a. In terms of interface dynamics, the presence of a small magnetic field breaking the $\mathbb{Z}_2$-symmetry should bring the system in the Kardar–Parisi–Zhang universality class [24]. It would here probably be desirable to have a microscopic model with $S \geqslant 3$ states available to investigate such cases with broken $\mathbb{Z}_2$-symmetry further.

4.2. Numerical integration

To verify the validity of the equations proposed in the previous sections, and to compare their behavior with the results of microscopic simulations, we have integrated them numerically, considering both deterministic cases and versions of the equations which are subjected to noise. It is well known that multiplicative noise prevents integration by naive Euler-forward discretization, but various methods have been recently proposed for overcoming this problem (see, e.g., [35, 39] and references therein). Unfortunately, a generalization of these step-split methods [35, 39] to systems of coupled stochastic differential equations with cross-correlated noises is still awaited, therefore in the case of equations (2) we limit our study to the deterministic variant of the equations. Slight dependencies on, e.g., system size and time-discretization in the numerical integration of these equations are here hard to eliminate due to limitations in computing time (we have used a standard Euler-forward scheme at a time-stepping of 0.01). Nevertheless, the results for the density of interfaces $n \sim 1 - \langle\phi^2\rangle$, reported in figure 1, are in good agreement with measurements from microscopic simulations, indicating a scaling exponent of about 0.45 to 0.5. They are also consistent with the snapshots generated from the microscopic dynamics (see figure 2) in which a curvature-driven coarsening is observed, even if domain boundaries are not completely smooth due to the interfacial roughening discussed above. This provides further evidence that noise cannot be expected to play a relevant role in the asymptotic coarsening dynamics of the voter model with intermediate states.

In the case of equation (6) the step-split method can be applied and both stochastic and deterministic versions of this reduced equation can hence be studied numerically. Split-step

methods for the stochastic case here rely on the fact that the Fokker–Planck equation for stochastic processes of the form $\dot{\phi} = \sqrt{\phi}\eta$ can be solved exactly, and that the resulting field distributions can subsequently be sampled. We here employ the scheme of [39], using the replacement $\sqrt{1-\phi^2} \to \Theta(\phi)\sqrt{1-\phi} + \Theta(-\phi)\sqrt{1+\phi}$ as described in [35]. The deterministic remainder of the equation is then integrated using a simple Euler-forward scheme. Typical time-steps used here are of the order of 0.25 [35, 39]. Results are shown in figure 1 for different values of the noise amplitude γ. Although the quantitative details of these numerical findings show dependencies on parameters such as system size or time-stepping, results are consistent with an exponent of 0.45 to 0.5 for the decay of the density of interfaces, and hence in good agreement with the above theoretical considerations and direct measurements from simulations of the microscopic dynamics of the three-state model. At variance with the analogous equation describing the standard voter model without intermediate states [35], and in agreement with the field theoretic prediction, the scaling behavior of the coarsening process is not significantly affected by noise amplitude γ in equation (9). In conclusion, numerical data obtained from integration of equations (2) and (6) corroborate the theoretical analysis and demonstrate the validity of the continuous approach to describe a long-time coarsening behavior of the microscopic voter model with intermediate states. The fact that both equations (2), and (6), (9) give an exponent $\delta \approx 0.45$ and that this value does not change much with the amplitude of the noise, leads one to suggest that the discrepancy with the theoretically expected value 0.5 of model A dynamics is probably due to finite size effects in the simulations, or to other numerical inaccuracies.

5. Conclusions

The voter model is one of the most studied microscopic models of interacting particle systems [6], and a paradigm for phase-ordering processes driven by interfacial noise. Recent developments in the study of non-equilibrium statistical physics have put forward a common paradigm to study coarsening with and without surface tension, which has led to the identification of the generalized voter (GV) universality class [22]. Voter models with modified spin–flip probability (i.e. breaking the linearity with the local field) have been proved to belong to this general class [5, 20, 21, 40]. In this paper we have shown that voter models with intermediate states fall into this class as well, and have studied their scaling behavior in detail. In particular, we observe algebraic domain ordering and the density of interfaces is found to decay with an exponent of 0.45 to 0.5.

We have derived a set of two coupled stochastic differential equations for appropriately defined fields, and have demonstrated that these equations faithfully describe the long-time large-scale behavior of the three-state voter model. Within a physically motivated approximation the two equations can be seen to reduce to one equation for a single field, and the latter equation in turn takes a form of a special case of the GV class. Moreover, the same algebraic coarsening is found in models with more than one intermediate state, and similar equations appear to hold for such cases with $S > 3$.

We have used a field theoretical approach to understand the universal properties of the model and have demonstrated that numerical results (obtained from both microscopic simulations and numerical integration of the effective equations) are consistent with this theoretical analysis. Interestingly we find that, due to the appearance of an effective surface tension induced by the presence of intermediate states, the domains coarsen like in the low-temperature phase of model A dynamics. In $d \leqslant 2$, the presence of interfacial noise induce interfacial roughening in agreement with the usual Edwards–Wilkinson scaling. The RG equations predict smooth interfaces in higher dimension.

It is hoped that our theoretical analysis will help to shed light on the phenomenology of this class of generalized voter models. These models have come to the attention of statistical physicists only recently, mostly due to their applications in the modeling of social systems. With a modest amount of imagination however, they can be expected to be applicable also in the study of kinetic reactions with intermediate unstable compounds and in other physical, biological or chemical non-equilibrium systems.

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References

[1] Bray A J 1994 *Adv. Phys.* **43** 357
[2] Gunton J D, San Miguel S and Sahni P S 1983 *Phase Transition and Critical Phenomena* Vol 3 ed C Domb and J L Lebowitz (New York: Academic)
[3] Hinrichsen H 2000 *Adv. Phys.* **49** 815
Ódor G 2004 *Rev. Mod. Phys.* **76** 663
[4] Castellano C, Fortunato S and Loreto V 2007 arXiv:0710.3256
[5] Dornic I, Chaté H, Chave J and Hinrichsen H 2001 *Phys. Rev. Lett.* **87** 045701
[6] Liggett T M 1985 *Interacting Particle Systems* (New York: Springer)
[7] Glauber R J 1963 *J. Math. Phys.* **4** 294
[8] Scheucher M and Spohn H 1988 *J. Stat. Phys.* **53** 279
[9] Peliti L 1986 *J. Phys. A: Math. Gen.* **19** L365
[10] Lee B P 1994 *J. Phys. A: Math. Gen.* **27** 2633
[11] Krapivsky P L 1992 *Phys. Rev.* A **45** 1067
[12] Frachebourg L and Krapivsky P L 1996 *Phys. Rev.* E **53** R3009
[13] de la Lama M S, Szendro I G, Iglesias J R and Wio H S 2006 *Eur. J. Phys.* B **51** 435–42
[14] Vazquez F, Krapivsky P L and Redner S 2003 *J. Phys. A: Math. Gen.* **36** L61–8
[15] Castelló X, Eguíluz V M and San Miguel M 2006 *New J. Phys.* **8** 308
[16] Baronchelli A, Dall'Asta L, Barrat A and Loreto V 2006 *Phys. Rev.* E **73** R015102
[17] Baronchelli A, Dall'Asta L, Barrat A and Loreto V 2007 *Phys. Rev.* E **76** 051102
[18] Dall'Asta L and Castellano C 2007 *Europhys. Lett.* **77** 60005
[19] Stark H-U, Tessone C J and Schweitzer F 2008 *Phys. Rev. Lett.* **101** 018701
[20] de Oliveira M J, F. Mendes J F and Santos M A 1993 *J. Phys. A: Math. Gen.* **26** 2317
[21] Drouffe J-M and Godréche C 1999 *J. Phys. A: Math. Gen.* **32** 249
[22] Hammal O Al, Chaté H, Dornic I and Muñoz M A 2005 *Phys. Rev. Lett.* **94** 230601
[23] Wang W S-Y and Minett J W 2005 *Trends Ecol. Evol.* **20** 263
[24] Barabási A-L and Stanley H E 1995 *Fractal Concepts in Surface Growth* (Cambridge: Cambridge University Press)
[25] Bray A J 1990 *Phys. Rev.* B **41** 6724
[26] Marro J and Dickman R 1999 *Non-Equilibrium Phase Transitions in Lattice Models* (Cambridge: Cambridge University Press)
[27] De Masi A, Ferrari P A and Lebowitz J L 1985 *Phys. Rev. Lett.* **55** 1947
[28] Cardy J L 1996 *Scaling and Renormalization In Statistical Physics* (Cambridge: Cambridge University Press)
[29] Cardy J L and Täuber U C 1996 *Phys. Rev. Lett.* **77** 4780
Cardy J L and Tauber U C 1998 *J. Stat. Phys.* **90** 1
[30] McKane A J and Newman T J 2004 *Phys. Rev.* E **70** 041902
[31] Reichenbach T, Mobilia M and Frey E 2007 *Phys. Rev. Lett.* **99** 238105
[32] Galla T and Dall'Asta L in preparation
[33] Dickman R and Tretyakov A Y 1995 *Phys. Rev.* E **52** 3218

[34] Canet L, Chaté H, Delamotte B, Dornic I and Muñoz M A 2005 *Phys. Rev. Lett.* **95** 100601
[35] Dornic I, Chaté H and Muñoz M A 2005 *Phys. Rev. Lett.* **94** 100601
[36] Jasnow D 1984 *Rep. Prog. Phys.* **47** 1059–132
[37] Parisi G 1988 *Statistical Field Theory* (New York: Addison-Wesley)
[38] Täuber U, Howard M and Vollmayr-Lee B 2005 *J. Phys. A: Math. Gen.* **38** R79–R131
[39] Moro E 2004 *Phys. Rev.* E **70** 045102
[40] Domany E and Kinzel W 1984 *Phys. Rev. Lett.* **53** 311